n-WEAK MODULE AMENABILITY OF TRIANGULAR BANACH ALGEBRAS

ABASALT BODAGHI* AND ALI JABBARI**

Abstract. Let $\mathcal{A}$, $\mathcal{B}$ be Banach $\mathfrak{A}$-modules with compatible actions and $\mathcal{M}$ be a left Banach $\mathcal{A}$-$\mathfrak{A}$-module and a right Banach $\mathcal{B}$-$\mathfrak{A}$-module. In the current paper, we study module amenability, n-weak module amenability and module Arens regularity of the triangular Banach algebra $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ (as an $\mathfrak{T} := \left\{ \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \mid \alpha \in \mathfrak{A} \right\}$-module). We employ these results to prove that for an inverse semigroup S with subsemigroup E of idempotents, the triangular Banach algebra $\mathcal{T}_0 = \begin{bmatrix} \ell^1(S) & \ell^1(S) \\ & \ell^1(S) \end{bmatrix}$ is permanently weakly module amenable (as an $\mathfrak{T}_0 = \begin{bmatrix} \ell^1(E) & \\ & \ell^1(E) \end{bmatrix}$-module). As an example, we show that $\mathcal{T}_0$ is $\mathfrak{T}_0$-module Arens regular if and only if the maximal group homomorphic image G_S of S is finite.

1. Introduction

The concept of module amenability for Banach algebras was initiated by Amini in [1]. The fundamental result was that the semigroup algebra $\ell^1(S)$ is module amenable as a Banach module on $\ell^1(E)$ if and only if S is amenable, where S is an inverse semigroup with subsemigroup E of idempotents. In fact he showed that Johnson's theorem [15] (for groups) holds for discrete inverse semigroups if the relevant module structure is taken into account. Amini and Bagha in [4] introduced the concept of weak amenability for Banach algebras showed that $\ell^1(S)$ is weakly $\ell^1(E)$-module amenable when S is a commutative inverse semigroup with the set of idempotents E (indeed this is true for inverse semigroups whose idempotents are central). Bodaghi et al. in [5] and [6] extended this result and showed that $\ell^1(S)$ is n-weakly module amenable as an $\ell^1(E)$-module (with trivial left action) when n is odd.

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Pourabbas and Nasrabadi investigated weak module amenability of a class of Banach algebras, called triangular Banach algebras in [18]. They considered the case where $\mathcal{A}$, $\mathcal{B}$ are unital Banach algebras (with $\mathfrak{A}$-module structure) and $\mathcal{M}$ is a unital Banach $\mathcal{A},\mathcal{B}$-module and showed that the corresponding triangular Banach algebra $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ is weakly module amenable (as an $\mathfrak{T} := \left\{ \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \mid \alpha \in \mathfrak{A} \right\}$-module) if and only if both $\mathcal{A}$ and $\mathcal{B}$ are weakly module amenable (as $\mathfrak{A}$-modules). This can be regarded as the module version of a result of Forrest and Marcoux [12, Corollary 3.5] (the case that $\mathcal{A}$ or $\mathcal{B}$ has a bounded approximate identity and $\mathcal{M}$ is essential was later proved by Medghalchi et al. in [16]). Also, they generalized this result to the case of $(2n-1)$-weak module amenability for $n \geq 1$ in [12, theorem 3.7].

The concept of Arens module regularity is introduced in [22] and modified in [19]. It is shown that $\mathcal{A} = \ell^1(S)$ is module Arens regular (as an $\ell^1(E)$-module) if and only if the group homomorphic image G_S of S is finite (see also [20]).

The motivation of writing this paper is *Example* 1 which shows that for the commutative inverse semigroup S with subsemigroup E of idempotents, the Banach algebra $\mathcal{T}_0 = \begin{bmatrix} \ell^1(S) & \ell^1(S) \\ & \ell^1(S) \end{bmatrix}$ is n-weakly module amenable (as an $\mathfrak{T}_0 := \left\{ \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \mid \alpha \in \ell^1(E) \right\}$-module) when $n \in \mathbb{N}$.

The paper is organized as follows: Section 2 is devoted to the study of module amenability of triangular Banach algebras. The main result of section 3 asserts that module Arens regularity of $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ is equivalent to that $\mathcal{A}$ and $\mathcal{B}$ are both module Arens regular and both act module regularly on $\mathcal{M}$. In section 4, we generalize some results of [12] and [16], and we show that the triangular Banach algebra $\mathcal{T}$ is $(2n-1)$-weakly module amenable (as an $\mathfrak{T}$-bimodule) if and only if $\mathcal{A}$ and $\mathcal{B}$ are $(2n-1)$-weakly module amenable (as Banach $\mathfrak{A}$-bimodules). In section 5, we prove that if $\mathcal{A}$ and $\mathcal{B}$ are $(2n)$-weakly module amenable, then the first module cohomology group of $\mathcal{T}$ with coefficients in $\mathcal{T}^{(2n)}$ is a quotient of a special set of module homomorphism from $\mathcal{M}$ to $\mathcal{M}^{(2n)}$. In section 6, we show that for a commutative inverse semigroup S with the set of idempotents E, the semigroup algebra $\ell^1(S)$ is n-weakly module amenable as an $\ell^1(E)$-module for all $n \in \mathbb{N}$. As a corollary, we show that $\mathcal{T}_0 = \begin{bmatrix} \ell^1(S) & \ell^1(S) \\ & \ell^1(S) \end{bmatrix}$ is permanently weakly module amenable (as an $\mathfrak{T}_0 = \begin{bmatrix} \ell^1(E) & \\ & \ell^1(E) \end{bmatrix}$-module). Finally, we show that$\mathcal{T}_0$ is module Arens regular if and only if $G_S = S/\approx$ is finite, where

$s \approx t$ whenever $\delta_s - \delta_t$ belongs to the closed linear span of the set $\{\delta_{set} - \delta_{st} : s, t \in S, e \in E\}$.

2. Module amenability

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \ (ab) \cdot \alpha = a(b \cdot \alpha) \tag{2.1}$$

for all $a, b \in \mathcal{A}, \alpha \in \mathfrak{A}$. Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \ (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \tag{2.2}$$

$$x \cdot (a \cdot \alpha) = (x \cdot a) \cdot \alpha, \ (a \cdot x) \cdot \alpha = a \cdot (x \cdot \alpha) \tag{2.3}$$

$$a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \ x \cdot (\alpha \cdot a) = (x \cdot \alpha) \cdot a \tag{2.4}$$

for all $a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X$. Then we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If

$$\alpha \cdot x = x \cdot \alpha \quad (\alpha \in \mathfrak{A}, x \in X)$$

then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. Moreover, if

$$a.x = x.a \qquad (a \in \mathcal{A}, x \in X),$$

then X is called a *bi-commutative* Banach $\mathcal{A}$-$\mathfrak{A}$-module.

If X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is X^*, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on X^* are defined as usual:

$$\langle f \cdot \alpha, x\rangle = \langle f, \alpha \cdot x\rangle, \ \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle,$$

$$\langle \alpha \cdot f, x\rangle = \langle f, x \cdot \alpha\rangle, \ \langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X, f \in X^*).$$

Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach $\mathcal{A}$-$\mathfrak{A}$-module, as we have not assumed the compatibility condition

$$a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b \quad (\alpha \in \mathfrak{A}, a, b \in \mathcal{A}).$$

If $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Also, if $\mathcal{A}$ is a commutative Banach algebra, then it is a bi-commutative $\mathcal{A}$-$\mathfrak{A}$-module. In these cases, $\mathcal{A}^{(n)}$, nth dual space of $\mathcal{A}$ is also a commutative or bi-commutative $\mathcal{A}$-$\mathfrak{A}$-module, respectively.

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules with compatible actions (2.1). Then a left $\mathfrak{A}$-module map is a mapping $T : \mathcal{A} \longrightarrow \mathcal{B}$ with $T(a \pm b) = T(a) \pm T(b)$ and $T(\alpha \cdot x) = \alpha \cdot T(x)$ for $a \in \mathcal{A}, b \in \mathcal{B}$, and $\alpha \in \mathfrak{A}$. A right or two-sided $\mathfrak{A}$-module map is defined similarly.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A ($\mathfrak{A}$-)*module derivation* is a bounded $\mathfrak{A}$-module map $D : \mathcal{A} \longrightarrow X$ satisfying

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in \mathcal{A}).$$

One should note that D is not necessarily linear, but its boundedness (defined as the existence of $M > 0$ such that $\|D(a)\| \leq M\|a\|$, for all $a \in \mathcal{A}$) still implies its continuity, as it preserves subtraction. When X is commutative, each $x \in X$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a \qquad (a \in \mathcal{A}).$$

These are called *inner* $\mathfrak{A}$-module derivations.

We use notations $\mathcal{Z}^1_{\mathfrak{A}}(\mathcal{A}, X)$ and $\mathcal{N}^1_{\mathfrak{A}}(\mathcal{A}, X)$ for the set of all module derivations and inner derivations from $\mathcal{A}$ to X, respectively. Also the quotient space $\mathcal{Z}^1_{\mathfrak{A}}(\mathcal{A}, X)/\mathcal{N}^1_{\mathfrak{A}}(\mathcal{A}, X)$ (which we call the first $\mathfrak{A}$-module cohomology group of $\mathcal{A}$ with coefficients in X) is denoted by $\mathcal{H}^1_{\mathfrak{A}}(\mathcal{A}, X)$. A Banach algebra $\mathcal{A}$ is module amenable if $\mathcal{H}^1_{\mathfrak{A}}(\mathcal{A}, X^*) = \{0\}$, for each commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X [1].

It is proved in [1, Proposition 2.5] that the homomorphic image of a module amenable Banach algebra under a continuous homomorphism with dense range is also a module amenable Banach algebra. This fact leads to the following result.

Lemma 2.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and $\mathcal{I}$ be a closed ideal in $\mathcal{A}$. Then module amenability of $\mathcal{A}$ implies module amenability of $\mathcal{A}/\mathcal{I}$.*

Proposition 2.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module, $\mathcal{I}$ be a closed ideal and $\mathfrak{A}$-submodule of $\mathcal{A}$. If $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are module amenable, then so is $\mathcal{A}$.*

Proof. Assume that X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module with compatible actions and $D : \mathcal{A} \longrightarrow X^*$ be a bounded module derivation. Since $\mathcal{I}$ is module amenable, there exists $f_1 \in X^*$ such that $D \mid_{\mathcal{I}} = D_{f_1}$. Thus, the map $\widetilde{D} = D - D_{f_1}$ vanishes on $\mathcal{I}$. This map induces a module derivation from $\mathcal{A}/\mathcal{I}$ into X^*, which we again denote by $\widetilde{D}$. Let Y be the closed linear span of

$$\{a \cdot x - y \cdot b \mid a, b \in \mathcal{I}, x, y \in X\},$$

in X. It follows immediately that Y is a closed $\mathcal{A}$-submodule and $\mathfrak{A}$-submodule of X, and so X/Y is a Banach $\mathcal{A}/\mathcal{I}$-$\mathfrak{A}$-module with compatible actions. Since $D \mid_{\mathcal{I}} = \{0\}$, we have $a \cdot \widetilde{D}(b) = \widetilde{D}(ab) - \widetilde{D}(a) \cdot b = 0$ for all $a \in \mathcal{I}$ and $b \in \mathcal{A}$. Similarly, $\widetilde{D}(b) \cdot a = 0$. This implies $\widetilde{D}(\mathcal{A}/\mathcal{I}) \subset Y^\perp = (X/Y)^*$. Due to module amenability of $\mathcal{A}/\mathcal{I}$, there is $f_2 \in Y^\perp \subset X^*$ such that $\widetilde{D} = D_{f_2}$. Consequently, $D = D_{f_1+f_2}$. □

Proposition 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules. Then $\mathcal{A} \oplus_{\ell^1} \mathcal{B}$, ℓ^1-direct sum of $\mathcal{A}$ and $\mathcal{B}$ is module amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are module amenable.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be module amenable. Since $\mathcal{B}$, the closed ideal of $\mathcal{A} \oplus_{\ell^1} \mathcal{B}$ and the quotient algebra $(\mathcal{A} \oplus_{\ell^1} \mathcal{B})/\mathcal{B} \cong \mathcal{A}$ are module amenable, $\mathcal{A} \oplus_{\ell^1} \mathcal{B}$ is module amenable by Proposition 2.1.

Conversely, assume that $\mathcal{A} \oplus_{\ell^1} \mathcal{B}$ is module amenable. Thus $(\mathcal{A} \oplus_{\ell^1} \mathcal{B})/\mathcal{A} \cong \mathcal{B}$ and $(\mathcal{A} \oplus_{\ell^1} \mathcal{B})/\mathcal{B} \cong \mathcal{A}$ are module amenable by Lemma 2.1. $\square$

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach $\mathfrak{A}$-bimodules with compatible actions (2.1) and $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions (2.2) and (2.3).

Let $J_{\mathcal{M}}$ be a subspace of $\mathcal{M}$ generated by elements of the form $a\cdot(\alpha\cdot x)-(a\cdot\alpha)\cdot x$ and $x \cdot (\alpha \cdot a) - (x \cdot \alpha) \cdot a$ for $\alpha \in \mathfrak{A}, a \in \mathcal{A}$ and $x \in \mathcal{M}$. One can see from (2.2) and (2.3) that $J_{\mathcal{M}}$ is an $\mathcal{A}$-submodule and $\mathfrak{A}$-submodule of $\mathcal{M}$. Note that the equalities (2.4) do not necessarily hold for $\mathcal{M}$. When $\mathcal{M} = \mathcal{A}$, $J_{\mathcal{A}}$ is in fact the closed ideal of $\mathcal{A}$ generated by elements of the form $(a \cdot \alpha)b - a(\alpha \cdot b)$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$ (see [5], [6] and [19]).

Let $\mathcal{T} = \left\{ \begin{bmatrix} a & m \\ & b \end{bmatrix} \mid a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M} \right\}$ be triangular Banach algebra equipped with the usual 2×2 matrix addition and formal multiplication. The norm on $\mathcal{T}$ is $\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \| = \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}} + \|m\|_{\mathcal{M}}$. Now, let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ be Banach $\mathfrak{A}$-bimodules, and let $\mathcal{M}$ be a Banach $\mathcal{A},\mathcal{B}$-module (left $\mathcal{A}$-module and right $\mathcal{B}$-module). Similar to [18], we consider the Banach algebra $\mathfrak{T} := \left\{ \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \mid \alpha \in \mathfrak{A} \right\}$. Then, the Banach algebra $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ with usual 2×2 matrix product is a $\mathfrak{T}$-bimodule. In fact, That is isomorphic to $\mathcal{A} \oplus_{\ell^1} \mathcal{M} \oplus_{\ell^1} \mathcal{B}$ as a Banach space and a Banach $\mathfrak{A}$-bimodule. The following result is a module version of [16, Theorem 4.2] and the proof is similar. However, we bring its proof.

Theorem 2.3. *Let $\mathcal{A}, \mathcal{B}$ be Banach $\mathfrak{A}$-modules and $\mathcal{M}$ be commutative Banach $\mathfrak{A}$-module. The triangular Banach algebra $\mathcal{T}$ is module amenable* (*as an $\mathfrak{T}$-bimodule*) *if and only if $\mathcal{A}$ and $\mathcal{B}$ are module amenable* (*as Banach $\mathfrak{A}$-bimodules*) *and $\mathcal{M} = 0$.*

Proof. First note that when $\mathcal{T}$ is $\mathfrak{T}$-bimodule it means that $\mathcal{A} \oplus_{\ell^1} \mathcal{M} \oplus_{\ell^1} \mathcal{B}$ is $\mathfrak{A}$-bimodule with the usual actions. Now, let $\mathcal{A}$ and $\mathcal{B}$ are module amenable and $\mathcal{M} = 0$. Then $\mathcal{T} = \begin{bmatrix} \mathcal{A} & 0 \\ & \mathcal{B} \end{bmatrix}$ is module amenable by Proposition 2.2.

Conversely, assume that $\mathcal{T}$ is $\mathfrak{T}$-module amenable. The Banach algebras $\begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ are closed ideals of $\mathcal{T}$, and thus $\mathcal{A}$ and $\mathcal{B}$ are $\mathfrak{A}$-module amenable by Lemma 2.1. Since $\mathcal{M}$ is complemented in $\mathcal{T}$, it is module amenable and since $\mathcal{M}$ is a commutative $\mathfrak{A}$-module, it has a bounded approximate identity by [1, Proposition 2.2], hence it should be zero. $\square$

3. Module Arens regularity

For given $\Gamma_1, \Gamma_2 \in \mathcal{A}^{**}$, by the Goldstein theorem there are nets $(a_{1,i})_i$ and $(a_{2,j})_j$ in $\mathcal{A}$ such that $\Gamma_1 = w^* - \lim_i a_{1,i}$ and $\Gamma_2 = w^* - \lim_j a_{2,j}$. Then we consider the first Arens product on $\mathcal{A}^{**}$ as follows:

$$\Gamma_1 \Box \Gamma_2 = w^* - \lim_i \lim_j a_{1,i} a_{2,j},$$

and similarly for any $\Psi_1, \Psi_2 \in \mathcal{B}^{**}$, there exist nets $(b_{1,i})_i$ and $(b_{2,j})_j$ such that the first Arens product on $\mathcal{B}$ is defined as follows:

$$\Psi_1 \Box \Psi_2 = w^* - \lim_i \lim_j b_{1,i} b_{2,j}.$$

We extend the actions of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{M}$ to actions of $\mathcal{A}^{**}$ and $\mathcal{B}^{**}$ on $\mathcal{M}^{**}$ via

$$\Gamma \Box \Pi = w^* - \lim_i \lim_k a_i \cdot x_k, \qquad \text{and} \qquad \Pi \Box \Psi = w^* - \lim_k \lim_j x_k \cdot b_j,$$

where $\Gamma = w^* - \lim_i a_i$, $\Psi = w^* - \lim_j b_j$, and $\Pi = w^* - \lim_k x_k$. We define the first Arens product on $\mathcal{T}^{**}$ in a natural way. Let $T_1 = \begin{bmatrix} \Gamma_1 & \Pi_1 \\ & \Psi_1 \end{bmatrix}, T_2 = \begin{bmatrix} \Gamma_2 & \Pi_2 \\ & \Psi_2 \end{bmatrix} \in \mathcal{T}^{**}$ so that $T_1 = w^* - \lim_i \begin{bmatrix} a_{1,i} & x_{1,i} \\ & b_{1,i} \end{bmatrix}$ and $T_2 = w^* - \lim_j \begin{bmatrix} a_{2,j} & x_{2,j} \\ & b_{2,j} \end{bmatrix}$. Then we have

$$\begin{aligned} T_1 \Box T_2 &= \begin{bmatrix} \Gamma_1 & \Pi_1 \\ & \Psi_1 \end{bmatrix} \Box \begin{bmatrix} \Gamma_2 & \Pi_2 \\ & \Psi_2 \end{bmatrix} \\ &= w^* - \lim_i \lim_j \begin{bmatrix} a_{1,i} & x_{1,i} \\ & b_{1,i} \end{bmatrix} \begin{bmatrix} a_{2,j} & x_{2,j} \\ & b_{2,j} \end{bmatrix} \\ &= w^* - \lim_i \lim_j \begin{bmatrix} a_{1,i} a_{2,j} & a_{1,i} x_{2,j} + x_{1,i} b_{2,j} \\ & b_{1,i} b_{2,j} \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_1 \Box \Gamma_2 & \Gamma_1 \Box \Pi_2 + \Pi_1 \Box \Psi_2 \\ & \Psi_1 \Box \Psi_2 \end{bmatrix}. \end{aligned}$$

Similarly, we consider the second Arens product on $\mathcal{A}^{**}$, $\mathcal{B}^{**}$ and module actions $\mathcal{A}^{**}$, $\mathcal{B}^{**}$ on $\mathcal{M}^{**}$ as follows:

$$\Gamma_1 \diamond \Gamma_2 = w^* - \lim_j \lim_i a_{1,i} a_{2,j}, \qquad \Psi_1 \diamond \Psi_2 = w^* - \lim_j \lim_i b_{1,i} b_{2,j},$$

and

$$\Gamma_1 \diamond \Pi = w^* - \lim_k \lim_i a_{1,i} x_k, \qquad \Pi \diamond \Psi_1 = w^* - \lim_j \lim_k x_k b_{1,j},$$

where $\Gamma_1, \Gamma_2 \in \mathcal{A}^{**}$, $\Psi_1, \Psi_2 \in \mathcal{B}^{**}$ and $\Pi \in \mathcal{M}^{**}$. Thus the second Arens product on $\mathcal{T}^{**}$ can be defined via

$$\begin{aligned} T_1 \diamond T_2 &= \begin{bmatrix} \Gamma_1 & \Pi_1 \\ & \Psi_1 \end{bmatrix} \diamond \begin{bmatrix} \Gamma_2 & \Pi_2 \\ & \Psi_2 \end{bmatrix} \\ &= w^* - \lim_j \lim_i \begin{bmatrix} a_{1,i} & x_{1,i} \\ & b_{1,i} \end{bmatrix} \begin{bmatrix} a_{2,j} & x_{2,j} \\ & b_{2,j} \end{bmatrix} \\ &= w^* - \lim_j \lim_i \begin{bmatrix} a_{1,i}a_{2,j} & a_{1,i}x_{2,j} + x_{1,i}b_{2,j} \\ & b_{1,i}b_{2,j} \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_1 \diamond \Gamma_2 & \Gamma_1 \diamond \Pi_2 + \Pi_1 \diamond \Psi_2 \\ & \Psi_1 \diamond \Psi_2 \end{bmatrix}. \end{aligned}$$

The concept of module Arens regularity for Banach algebras is defined in [22]. A Banach algebra $\mathcal{A}$ is module Arens regular as a Banach $\mathfrak{A}$-module if and only if $\Gamma_1 \square \Gamma_2 - \Gamma_1 \diamond \Gamma_2 \in J_{\mathcal{A}}^{\perp\perp}$, for every $\Gamma_1, \Gamma_2 \in \mathcal{A}^{**}$ (see [22, Theorem 2.2] and [3, Theorem 3.3]).

We say that the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ act *module regularly* on $\mathcal{M}$ if for each $\Gamma \in \mathcal{A}^{**}$, $\Psi \in \mathcal{B}^{**}$ and $\Pi \in \mathcal{M}^{**}$ we have

$$\Gamma \square \Pi - \Gamma \diamond \Pi \in J_{\mathcal{M}}^{\perp\perp}, \quad \Pi \square \Psi - \Pi \diamond \Psi \in J_{\mathcal{M}}^{\perp\perp}.$$

Recall that the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ act regularly on $\mathcal{A},\mathcal{B}$-module $\mathcal{M}$ if for every $\Gamma \in \mathcal{A}^{**}$, $\Psi \in \mathcal{B}^{**}$ and $\Pi \in \mathcal{M}^{**}$ we have $\Gamma \square \Pi = \Gamma \diamond \Pi$ and $\Pi \square \Psi = \Pi \diamond \Psi$ [13]. It follows from the above that $\mathcal{A}$ and $\mathcal{B}$ act module regularly on $\mathcal{M}$ if and only if $\mathcal{A}/J_{\mathcal{A}}$ and $\mathcal{B}/J_{\mathcal{B}}$ act regularly on $\mathcal{M}/J_{\mathcal{M}}$. Indeed,

$$\begin{aligned} \Gamma \square \Pi - \Gamma \diamond \Pi \in J_{\mathcal{M}}^{\perp\perp} &\Longleftrightarrow \Gamma \square \Pi + J_{\mathcal{M}}^{\perp\perp} = \Gamma \diamond \Pi + J_{\mathcal{M}}^{\perp\perp} \\ &\Longleftrightarrow (\Gamma + J_{\mathcal{A}}^{\perp\perp}) \square (\Pi + J_{\mathcal{M}}^{\perp\perp}) = (\Gamma + J_{\mathcal{A}}^{\perp\perp}) \diamond (\Pi + J_{\mathcal{M}}^{\perp\perp}). \end{aligned}$$

Similarly, $\Pi \square \Psi - \Pi \diamond \Psi \in J_{\mathcal{M}}^{\perp\perp}$ if and only if $(\Pi + J_{\mathcal{M}}^{\perp\perp}) \square (\Psi + J_{\mathcal{B}}^{\perp\perp}) = (\Pi + J_{\mathcal{M}}^{\perp\perp}) \diamond (\Psi + J_{\mathcal{B}}^{\perp\perp})$. Similarly we can show that $J_{\mathcal{T}} = \begin{bmatrix} J_{\mathcal{A}} & J_{\mathcal{M}} \\ & J_{\mathcal{B}} \end{bmatrix}$ and thus $\begin{bmatrix} J_{\mathcal{A}}^{\perp} & J_{\mathcal{M}}^{\perp} \\ & J_{\mathcal{B}}^{\perp} \end{bmatrix} \subseteq J_{\mathcal{T}}^{\perp}$. It is shown in [6, Lemma 3.1] that $J_{\mathcal{A}}$ and $J_{\mathcal{B}}$ are the closed subspace of $\mathcal{A}$ an $\mathcal{B}$ respectively. It is easy to check that $J_{\mathcal{M}}$ is also a closed subspace of $\mathcal{M}$. Hence $J_{\mathcal{A}}, J_{\mathcal{B}}$ and $J_{\mathcal{M}}$ are weak*-dense in $J_{\mathcal{A}}^{\perp\perp}, J_{\mathcal{B}}^{\perp\perp}$ and $J_{\mathcal{M}}^{\perp\perp}$ respectively by [9, Theorem A.3.47]. Hence $J_{\mathcal{A}} \oplus_{\ell^1} J_{\mathcal{M}} \oplus_{\ell^1} J_{\mathcal{B}}$ is weak*-dense in $J_{\mathcal{A}}^{\perp\perp} \oplus_{\ell^1} J_{\mathcal{M}}^{\perp\perp} \oplus_{\ell^1} J_{\mathcal{B}}^{\perp\perp}$. On the other hand, $J_{\mathcal{A}} \oplus_{\ell^1} J_{\mathcal{M}} \oplus_{\ell^1} J_{\mathcal{B}}$ is closed subspace of $\mathcal{A} \oplus_{\ell^1} \mathcal{M} \oplus_{\ell^1} \mathcal{B}$ and so it is weak*-dense in $J_{\mathcal{T}}^{\perp\perp}$. Therefore $J_{\mathcal{T}}^{\perp\perp} = \begin{bmatrix} J_{\mathcal{A}}^{\perp\perp} & J_{\mathcal{M}}^{\perp\perp} \\ & J_{\mathcal{B}}^{\perp\perp} \end{bmatrix}$. Summing up:

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules. The triangular Banach algebra $\mathcal{T}$ is module Arens regular (as an $\mathfrak{T}$-bimodule) if and only if $\mathcal{A}$ and $\mathcal{B}$*

are module Arens regular (as Banach $\mathfrak{A}$-bimodules) and $\mathcal{A}$ and $\mathcal{B}$ act module regularly on $\mathcal{M}$.

4. (2n-1)-Weak Module Amenability

We start this section with the definition of n-weak module amenability which is introduced in [6]. When $\mathcal{A}$ is a commutative $\mathfrak{A}$-bimodule, we have $J = \{0\}$ and $\mathcal{A}$ is a commutative $\mathcal{A}$-$\mathfrak{A}$-module. In this case, the following definition (for $n = 1$) coincides with the definition of weak module amenability in [4]. If $\mathcal{A}$ is a commutative Banach algebra and a commutative $\mathfrak{A}$-bimodule with compatible actions, then $\mathcal{A}$ is a bi-commutative $\mathcal{A}$-$\mathfrak{A}$-module. In this case, for each bi-commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, all bounded module derivation from $\mathcal{A}$ into X are zero (see the next lemma) and we get the definition of weak module amenability for commutative Banach algebras as in [4].

Definition 4.1. Let $\mathcal{A}$ be a Banach algebra, $n \in \mathbb{N}$. Then $\mathcal{A}$ is called *n-weakly module amenable* (as an $\mathfrak{A}$-module) if $(\mathcal{A}/J)^{(n)}$ is a commutative $\mathcal{A}$-$\mathfrak{A}$-module, and each module derivation from $D : \mathcal{A} \longrightarrow (\mathcal{A}/J)^{(n)}$ is inner; that is, $D(a) = a \cdot y - y \cdot a =: D_y(a)$ for some $y \in (\mathcal{A}/J)^{(n)}$ and each $a \in A$. Also $\mathcal{A}$ is called *weakly module amenable* if it is 1-weakly module amenable and *permanently weakly module amenable* if it is n-weakly module amenable for each $n \in \mathbb{N}$.

Lemma 4.1. *Let $\mathcal{A}$ be a essential bi-commutative $\mathcal{A}$-$\mathfrak{A}$-module. Then $\mathcal{A}$ is weakly module amenable (as an $\mathfrak{A}$-module) if and only if for each bi-commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, all bounded module derivation from $\mathcal{A}$ into X are zero.*

Proof. We follow the standard argument in [9, Theorem 2.8.63]. Assume that there exists $D \in \mathcal{Z}_{\mathfrak{A}}(\mathcal{A}, X)$ with $D \neq 0$. Since $\overline{\mathcal{A}^2} = \mathcal{A}$, there exists $a_0 \in \mathcal{A}$ such that $D(a_0^2) \neq 0$. We have $a_0 \cdot D(a_0) \neq 0$ and thus $f \in X^*$ with $f(a_0 \cdot D(a_0)) = 1$. Set $R : X \longrightarrow \mathcal{A}^*$ defined by $R(x)(a) = f(a \cdot x)$ where $a \in \mathcal{A}, x \in X$. It is easy to check that $R \circ D \in \mathcal{Z}_{\mathfrak{A}}(\mathcal{A}, \mathcal{A}^*)$. We get $\langle R \circ D(a_0), a_0\rangle = \langle f, a_0 \cdot D(a_0)\rangle = 1$, and so $R \circ D \neq 0$. This shows that $\mathcal{A}$ is not weakly module amenable. The converse is clear. $\square$

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ be as in the previous section. If these are commutative Banach $\mathfrak{A}$-modules, then the corresponding triangular Banach algebra $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ is a commutative $\mathfrak{T}$-module in which $\mathfrak{T} := \left\{ \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \mid \alpha \in \mathfrak{A} \right\}$. Also $\mathcal{T}$ is isomorphic to $\mathcal{A} \oplus_{\ell^1} \mathcal{M} \oplus_{\ell^1} \mathcal{B}$ as a Banach $\mathfrak{T}$-module and a Banach $\mathfrak{A}$-module, respectively. Therefore $\mathcal{T}^{(2n-1)} \simeq \mathcal{A}^{(2n-1)} \oplus_{\ell^1} \mathcal{M}^{(2n-1)} \oplus_{\ell^1} \mathcal{B}^{(2n-1)}$, while $\mathcal{T}^{(2n)} \simeq \mathcal{A}^{(2n)} \oplus_{\ell^\infty} \mathcal{M}^{(2n)} \oplus_{\ell^\infty} \mathcal{B}^{(2n)}$ in which $\mathcal{T}^{(n)}$ is Banach $\mathfrak{T}$-module and $\mathcal{A}^{(2n-1)} \oplus_{\ell^1} \mathcal{M}^{(2n-1)} \oplus_{\ell^1} \mathcal{B}^{(2n-1)}$, $\mathcal{A}^{(2n)} \oplus_{\ell^\infty} \mathcal{M}^{(2n)} \oplus_{\ell^\infty} \mathcal{B}^{(2n)}$ are Banach $\mathfrak{A}$-modules. Suppose that $\mathfrak{t} = \begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$and $\theta = \begin{bmatrix} f & \lambda \\ & g \end{bmatrix} \in \mathcal{T}^*$. Then the

pairing of $\mathcal{T}^*$ and $\mathcal{T}$ is given by $\theta(\mathfrak{t}) = f(a) + \lambda(m) + g(b)$. Indeed, it is easy to check that the module actions $\mathcal{T}$ on $\mathcal{T}^*$ are as follows:

$$\mathfrak{t} \cdot \theta = \begin{bmatrix} a \cdot f + m \cdot \lambda & b \cdot \lambda \\ & b \cdot g \end{bmatrix} \quad \text{and} \quad \theta \cdot \mathfrak{t} = \begin{bmatrix} f \cdot a & \lambda \cdot a \\ & \lambda \cdot m + g \cdot b \end{bmatrix}.$$

We remove the dot for simplicity. This process may be repeated to define the actions of $\mathcal{T}$ on $\mathcal{T}^{(n)}$ as follows:

$$\begin{bmatrix} a & m \\ & b \end{bmatrix} \cdot \begin{bmatrix} \lambda & \gamma \\ & \mu \end{bmatrix} = \begin{bmatrix} a\lambda & a\gamma + m\mu \\ & b\mu \end{bmatrix},$$

$$\begin{bmatrix} \lambda & \gamma \\ & \mu \end{bmatrix} \cdot \begin{bmatrix} a & m \\ & b \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda m + \gamma b \\ & \mu b \end{bmatrix}$$

and

$$\begin{bmatrix} a & m \\ & b \end{bmatrix} \cdot \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} = \begin{bmatrix} a\phi + m\varphi & b\varphi \\ & b\psi \end{bmatrix},$$

$$\begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \cdot \begin{bmatrix} a & m \\ & b \end{bmatrix} = \begin{bmatrix} \phi a & \varphi a \\ & \psi b + \varphi m \end{bmatrix},$$

for every $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$, $\begin{bmatrix} \lambda & \gamma \\ & \mu \end{bmatrix} \in \mathcal{T}^{(2n)}$ and $\begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \in \mathcal{T}^{(2n-1)}$.

Henceforth, we assume that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ are commutative Banach $\mathfrak{A}$-modules and thus $\mathcal{T}^{(2n-1)}$ and $\mathcal{T}^{(2n)}$ become commutative Banach $\mathfrak{T}$-modules for any $n \in \mathbb{N}$.

The following two lemmas are proved similar to Lemmas 1.1 and 1.2 in [18].

Lemma 4.2. *The $\mathfrak{T}$-module map $D : \mathcal{T} \longrightarrow \mathcal{T}^{(2n-1)}$ is a module derivation if and only if there exist module derivations $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^{(2n-1)}$, $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}^{(2n-1)}$ and $\gamma \in \mathcal{M}^{(2n-1)}$ such that*

$$D\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\Big) = \begin{bmatrix} D_{\mathcal{A}}(a) - m\gamma & \gamma a - b\gamma \\ & D_{\mathcal{B}}(b) + \gamma m \end{bmatrix}, \tag{4.1}$$

for every $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$.

Lemma 4.3. *Let $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^{(2n-1)}$ and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}^{(2n-1)}$ be bounded module derivations. Then $D_{\mathcal{AB}} : \mathcal{T} \longrightarrow \mathcal{T}^{(2n-1)}$ defined via*

$$\begin{bmatrix} a & m \\ & b \end{bmatrix} \longmapsto \begin{bmatrix} D_{\mathcal{A}}(a) & \\ & D_{\mathcal{B}}(b) \end{bmatrix}, \tag{4.2}$$

is a bounded module derivation. Furthermore, $D_{\mathcal{AB}}$ is inner if and only if $D_{\mathcal{A}}$ and $D_{\mathcal{B}}$ are inner.

Using Lemmas 4.2 and 4.3, and similar to [18, Theorem 2.1], we have the following result (see also [16, Theorem 2.1]) .

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras, and $\mathcal{M}$ be a unital $\mathcal{A},\mathcal{B}$-module. Then*

$$\mathcal{H}^1_{\mathfrak{T}}(\mathcal{T},\mathcal{T}^{(2n-1)}) \simeq \mathcal{H}^1_{\mathfrak{A}}(\mathcal{A},\mathcal{A}^{(2n-1)}) \oplus \mathcal{H}^1_{\mathfrak{A}}(\mathcal{B},\mathcal{B}^{(2n-1)}). \tag{4.3}$$

In particular, the triangular Banach algebra $\mathcal{T}$ is $(2n-1)$-weakly module amenable (as an $\mathfrak{T}$-bimodule) if and only if $\mathcal{A}$ and $\mathcal{B}$ are $(2n-1)$-weakly module amenable (as Banach $\mathfrak{A}$-bimodules).

A Banach $\mathcal{A},\mathcal{B}$-module $\mathcal{M}$ is said to be non-degenerate if $\mathcal{A}m = \{0\}$ implies that $m = 0$, and $m\mathcal{B} = \{0\}$ implies that $m = 0$ for every $m \in \mathcal{M}$. If the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ have bounded approximate identity and $\mathcal{M}$ is essential, then $\mathcal{M}$ is a non-degenerated $\mathcal{A},\mathcal{B}$-module. Also when $\mathcal{M}$ is essential, then $\mathcal{M}^*$ is a non-degenerate Banach $\mathcal{A},\mathcal{B}$-module. Although in the following Proposition $\mathcal{A}$ and $\mathcal{B}$ are not unital but still the conclusion of Lemma 4.2 holds. In fact we obtain the same result with different conditions.

Proposition 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and $\mathcal{M}$ be Banach $\mathcal{A},\mathcal{B}$-module. Suppose that $\mathcal{A}$ possess a bounded approximate identity, $\mathcal{A}^{(2n-1)},\mathcal{B}^{(2n-1)}$ and $\mathcal{M}^{(2n-1)}$ are non-degenerate. Then the $\mathfrak{T}$-module map $D : \mathcal{T} \longrightarrow \mathcal{T}^{(2n-1)}$ is module derivation if and only if there exist module derivations $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^{(2n-1)}$, $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}^{(2n-1)}$ and $\gamma \in \mathcal{M}^{(2n-1)}$ such that*

$$D\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\Big) = \begin{bmatrix} D_{\mathcal{A}}(a) - m\gamma & \gamma a - b\gamma \\ & D_{\mathcal{B}}(b) + \gamma m \end{bmatrix}, \tag{4.4}$$

for every $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$.

Proof. Let $D : \mathcal{T} \longrightarrow \mathcal{T}^{(2n-1)}$ be a continuous module derivation. Define $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^{(2n-1)}$ by $D_{\mathcal{A}}(a) = \pi_{\mathcal{A}}(D\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big))$, and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}^{(2n-1)}$ via $D_{\mathcal{B}}(b) = \pi_{\mathcal{B}}(D\Big(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\Big))$. Obviously these maps are $\mathfrak{A}$-module maps, and are module derivations by [16, Lemma 2.3]. Let $(e_\alpha)_{\alpha\in\Lambda}$ be a bounded approximate identity of $\mathcal{A}$, and let $D\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} D_{\mathcal{A}}(a) & \eta \\ & \mu \end{bmatrix}$ for an arbitrary and fixed

$a \in \mathcal{A}$. Then

$$\begin{aligned} D\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big) &= D\Big(\begin{bmatrix} \lim_\alpha e_\alpha a & 0 \\ & 0 \end{bmatrix}\Big) = D\Big(\lim_\alpha \begin{bmatrix} e_\alpha & 0 \\ & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big) \\ &= \lim_\alpha \Big(\begin{bmatrix} e_\alpha & 0 \\ & 0 \end{bmatrix} \cdot \begin{bmatrix} D_{\mathcal{A}}(a) & \eta \\ & \mu \end{bmatrix} \\ &+ \begin{bmatrix} D_{\mathcal{A}}(e_\alpha) & \gamma \\ & \theta \end{bmatrix} \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big) \\ &= \begin{bmatrix} \lim_\alpha D_{\mathcal{A}}(ae_\alpha) & \gamma a \\ & 0 \end{bmatrix} = \begin{bmatrix} D_{\mathcal{A}}(a) & \gamma a \\ & 0 \end{bmatrix}. \end{aligned} \tag{4.5}$$

Similarly, consider $b \in \mathcal{B}$ such that $D\Big(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\Big) = \begin{bmatrix} \theta & \eta \\ & D_{\mathcal{B}}(b) \end{bmatrix}$. Since $\mathcal{M}^{(2n-1)}$ and $\mathcal{A}^{(2n-1)}$ are non-degenerate (see [16, Proposition 2.5]), we have

$$D\Big(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\Big) = \begin{bmatrix} 0 & -b\gamma \\ & D_{\mathcal{B}}(b) \end{bmatrix}. \tag{4.6}$$

Also for every $m \in \mathcal{M}$ we have

$$D\Big(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} -m\gamma & 0 \\ & \gamma m \end{bmatrix}. \tag{4.7}$$

Now, from (4.5), (4.6) and (4.7), we get (4.4), and this completes the proof. □

Now we are ready to prove the main theorem of this section (see also the proof of [18, Theorem 2.1]).

Theorem 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ both have bounded approximate identity, and let $\mathcal{M}$ be non-degenerate. Then for every $n \geq 1$ we have*

$$\mathcal{H}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n-1)}) \simeq \mathcal{H}^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{A}^{(2n-1)}) \oplus \mathcal{H}^1_{\mathfrak{A}}(\mathcal{B}, \mathcal{B}^{(2n-1)}). \tag{4.8}$$

Furthermore, the corresponding triangular Banach algebra $\mathcal{T}\begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ is $(2n-1)$-weakly module amenable (as an $\mathfrak{T}$-bimodule) if and only if $\mathcal{A}$ and $\mathcal{B}$ are $(2n-1)$-weakly module amenable (as Banach $\mathfrak{A}$-bimodules).

Proof. Suppose that $D : \mathcal{T} \longrightarrow \mathcal{T}^{(2n-1)}$ is a continuous module derivation. Proposition 4.2 shows that there are continuous module derivations $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^{(2n-1)}$, $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}^{(2n-1)}$ and $\gamma \in \mathcal{M}^{(2n-1)}$ such that

$$D\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\Big) = \begin{bmatrix} D_{\mathcal{A}}(a) - m\gamma & \gamma a - b\gamma \\ & D_{\mathcal{B}}(b) + \gamma m \end{bmatrix}, \tag{4.9}$$

for every $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$. Define $\Psi : \mathcal{Z}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n-1)}) \longrightarrow \mathcal{H}^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{A}^{(2n-1)}) \oplus \mathcal{H}^1_{\mathfrak{T}}(\mathcal{B}, \mathcal{B}^{(2n-1)})$ by

$$\Psi(D) = (D_{\mathcal{A}} + \mathcal{N}^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{A}^{(2n-1)}), D_{\mathcal{B}} + \mathcal{N}^1_{\mathfrak{A}}(\mathcal{B}, \mathcal{B}^{(2n-1)})). \tag{4.10}$$

Let's show that Ψ is a linear map. If (e_α) is a bounded approximate identity for $\mathcal{A}$, then for given $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} \langle \lambda a', D_{\mathcal{A}}(a)\rangle &= \langle \lim_\alpha \lambda a' e_\alpha, D_{\mathcal{A}}(a)\rangle = \lim_\alpha \langle a', \lambda e_\alpha D_{\mathcal{A}}(a)\rangle \\ &= \langle \lim_\alpha a' e_\alpha, \lambda D_{\mathcal{A}}(a)\rangle = \langle a', \lambda D_{\mathcal{A}}(a)\rangle \quad (a, a' \in \mathcal{A}) \end{aligned} \tag{4.11}$$

and similarly we have

$$\langle \lambda b', D_{\mathcal{B}}(b)\rangle = \langle b', \lambda D_{\mathcal{B}}(b)\rangle \qquad (b, b' \in \mathcal{B}). \tag{4.12}$$

Now, by applying relations (4.11) and (4.12) for every $T_1 = \begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}$ and $T_2 = \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}$ in $\mathcal{T}$ we get

$$\begin{aligned} \langle (\lambda T_2), D(T_1)\rangle &= D\Big(\begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}\Big)\Big(\begin{bmatrix} \lambda a_2 & \lambda m_2 \\ & \lambda b_2 \end{bmatrix}\Big) \\ &= \begin{bmatrix} D_{\mathcal{A}}(a_1) - m_1\gamma & \gamma a_1 - b_1\gamma \\ & D_{\mathcal{B}}(b_1) + \gamma m_1 \end{bmatrix}\Big(\begin{bmatrix} \lambda a_2 & \lambda m_2 \\ & \lambda b_2 \end{bmatrix}\Big) \\ &= D_{\mathcal{A}}(a_1)(\lambda a_2) - m_1\gamma(\lambda a_2) + a_1\gamma(\lambda b_2) \\ &- b_1\gamma(\lambda m_2) + D_{\mathcal{B}}(b)(\lambda b_2) + m\gamma(\lambda b_2) \\ &= \lambda D_{\mathcal{A}}(a_1)(a_2) - \lambda m_1\gamma(a_2) + \lambda a_1\gamma(b_2) \\ &- \lambda b_1\gamma(m_2) + \lambda D_{\mathcal{B}}(b)(b_2) + \lambda m\gamma(b_2) \\ &= \begin{bmatrix} \lambda D_{\mathcal{A}}(a_1) - \lambda m_1\gamma & \lambda\gamma a_1 - \lambda b_1\gamma \\ & \lambda D_{\mathcal{B}}(b_1) + \lambda\gamma m_1 \end{bmatrix}\Big(\begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}\Big) \\ &= \lambda D(T_1)(T_2). \end{aligned} \tag{4.13}$$

Thus $\Psi(\lambda D) = \lambda\Psi(D)$. Obviously, $\Psi(D_1 + D_2) = \Psi(D_1) + \Psi(D_2)$ for all $D_1, D_2 \in \mathcal{Z}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n-1)})$. Hence, Ψ is a linear map. Now, assume that $D_{\mathcal{A}} \in \mathcal{Z}^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{A}^{(2n-1)})$ and $D_{\mathcal{B}} \in \mathcal{Z}^1_{\mathfrak{A}}(\mathcal{B}, \mathcal{B}^{(2n-1)})$. Then Lemma 4.3 implies that there exists a module derivation $D_{\mathcal{AB}}$ such that

$$\Psi(D_{\mathcal{AB}}) = (D_{\mathcal{A}} + \mathcal{N}^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{A}^{(2n-1)}), D_{\mathcal{B}} + \mathcal{N}^1_{\mathfrak{A}}(\mathcal{B}, \mathcal{B}^{(2n-1)})),$$

and this indicates that Ψ is surjective. If $D \in \ker\Psi$, then $\Psi(D) = 0$ and thus $D_{\mathcal{A}} \in \mathcal{N}^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{A}^{(2n-1)})$ and $D_{\mathcal{B}} \in \mathcal{N}^1_{\mathfrak{A}}(\mathcal{B}, \mathcal{B}^{(2n-1)})$. Since $D_{\mathcal{A}}$ and $D_{\mathcal{B}}$ are inner, $D_{\mathcal{AB}}$ is an inner module derivation by Lemma 4.3. Therefore we can write

$$\begin{bmatrix} D_{\mathcal{A}}(a) - m\gamma & \gamma a - b\gamma \\ & D_{\mathcal{B}}(b) + \gamma m \end{bmatrix} = \begin{bmatrix} D_{\mathcal{A}}(a) & \\ & D_{\mathcal{B}}(b) \end{bmatrix} + \begin{bmatrix} -m\gamma & \gamma a - b\gamma \\ & \gamma m \end{bmatrix}.$$

The above equality shows that D is inner. Thus, $\ker\Psi \subseteq \mathcal{N}^1_{\mathfrak{X}}(\mathcal{T},\mathcal{T}^{(2n-1)})$. On the other hand, if $D \in \mathcal{N}^1_{\mathfrak{X}}(\mathcal{T},\mathcal{T}^{(2n-1)})$, then $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^{(2n-1)}$ defined by $D_{\mathcal{A}}(a) = \pi_{\mathcal{A}}(D\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big))$ and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}^{(2n-1)}$ defined by $D_{\mathcal{B}}(b) = \pi_{\mathcal{B}}(D\Big(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\Big))$ are inner module derivations. Therefore $\Psi(D) = 0$, and so $\mathcal{N}^1_{\mathfrak{X}}(\mathcal{T},\mathcal{T}^{(2n-1)}) \subseteq \ker\Psi$. Finally, we have

$$\begin{aligned} \mathcal{H}^1_{\mathfrak{X}}(\mathcal{T},\mathcal{T}^{(2n-1)}) &= \mathcal{Z}^1_{\mathfrak{X}}(\mathcal{T},\mathcal{T}^{(2n-1)})/\mathcal{N}^1_{\mathfrak{X}}(\mathcal{T},\mathcal{T}^{(2n-1)}) \\ &= \mathcal{Z}^1_{\mathfrak{X}}(\mathcal{T},\mathcal{T}^{(2n-1)})/\ker\Psi \\ &\simeq \mathrm{Im}\Psi = \mathcal{H}^1_{\mathfrak{A}}(\mathcal{A},\mathcal{A}^{(2n-1)}) \oplus \mathcal{H}^1_{\mathfrak{A}}(\mathcal{B},\mathcal{B}^{(2n-1)}). \end{aligned}$$

□

5. (2n)-Weak Module Amenability

As it is seen in the previous section, $(2n-1)$-weak module amenability of a triangular Banach algebra $\mathcal{T}$ depends on $(2n-1)$-weak module amenability of Banach algebras $\mathcal{A}$ and $\mathcal{B}$ while this fails to be true in the even case in general. We need the following lemma which is analogous to [13, Proposition 3.9] in the module case. We include the proof.

Lemma 5.1. *Let $D : \mathcal{T} \longrightarrow \mathcal{T}^{(2n)}$ be a continuous module derivation. Then there exist $\gamma \in \mathcal{M}^{(2n)}$, continuous module derivations $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^{(2n)}$, $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}^{(2n)}$ and a continuous $\mathfrak{A}$-module map $\rho : \mathcal{M} \longrightarrow \mathcal{M}^{(2n)}$ such that*

(i) $D\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} D_{\mathcal{A}}(a) & a\cdot\gamma \\ & 0 \end{bmatrix}, \quad (a \in \mathcal{A})$;

(ii) $D\Big(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\Big) = \begin{bmatrix} 0 & -\gamma\cdot b \\ & D_{\mathcal{B}}(b) \end{bmatrix}, \quad (b \in \mathcal{B})$;

(iii) $D\Big(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} 0 & \rho(m) \\ & 0 \end{bmatrix}, \quad (m \in \mathcal{M})$;

(iv) $\rho(a\cdot m) = D_{\mathcal{A}}(a)\cdot m + a\cdot\rho(m), \quad (a \in \mathcal{A}, m \in \mathcal{M})$;

(v) $\rho(m\cdot b) = \rho(m)\cdot b + m\cdot D_{\mathcal{B}}(b) \quad (a \in \mathcal{A}, m \in \mathcal{M})$;

(vi) *If $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^{(2n)}$ and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}^{(2n)}$ are continuous module derivations and $\rho_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}^{(2n)}$ is a continuous $\mathfrak{A}$-module map that satisfies* (iv) *and* (v)*. Then $D : \mathcal{T} \to \mathcal{T}^{(2n)}$ defined by $\begin{bmatrix} a & m \\ & b \end{bmatrix} \longmapsto \begin{bmatrix} D_{\mathcal{A}}(a) & \rho_{\mathcal{M}}(m) \\ & D_{\mathcal{B}}(b) \end{bmatrix}$ is a continuous module derivation.*

Proof. In the light of [13, Proposition 3.9], we just show that these maps are module maps. Let D be a $\mathfrak{T}$-module map. Then

$$\begin{aligned}
\langle a_2, \alpha \cdot D_{\mathcal{A}}(a_1)\rangle &= \langle a_2 \cdot \alpha, D_{\mathcal{A}}(a_1)\rangle = \langle a_2 \cdot \alpha, D_{\mathcal{A}}(a_1)\rangle + \langle 0, a_1 \cdot \gamma\rangle \\
&= \langle \begin{bmatrix} a_2 \cdot \alpha & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} D_{\mathcal{A}}(a_1) & a_1 \cdot \rho \\ & 0 \end{bmatrix}\rangle \\
&= \langle \begin{bmatrix} a_2 & 0 \\ & 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha & 0 \\ & \alpha \end{bmatrix}, D\Big(\begin{bmatrix} a_1 & 0 \\ & 0 \end{bmatrix}\Big)\rangle \\
&= \langle \begin{bmatrix} a_2 & 0 \\ & 0 \end{bmatrix}, D\Big(\begin{bmatrix} \alpha & 0 \\ & \alpha \end{bmatrix} \cdot \begin{bmatrix} a_1 & 0 \\ & 0 \end{bmatrix}\Big)\rangle \\
&= \langle \begin{bmatrix} a_2 & 0 \\ & 0 \end{bmatrix}, D\Big(\begin{bmatrix} \alpha \cdot a_1 & 0 \\ & 0 \end{bmatrix}\Big)\rangle \\
&= \langle \begin{bmatrix} a_2 & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} D_{\mathcal{A}}(\alpha a_1) & \alpha \cdot a_1 \cdot \rho \\ & 0 \end{bmatrix}\rangle \\
&= \langle a_2, D_{\mathcal{A}}(\alpha \cdot a_1)\rangle + \langle 0, \alpha \cdot a_1 \cdot \gamma\rangle = \langle a_2, D_{\mathcal{A}}(\alpha \cdot a_1)\rangle,
\end{aligned}$$

for all $a_1, a_2 \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. This means that $D_{\mathcal{A}}(\alpha \cdot a) = \alpha \cdot D_{\mathcal{A}}(a)$, for all $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. Similarly we can show that $D_{\mathcal{A}}(a \cdot \alpha) = D_{\mathcal{A}}(a) \cdot \alpha$ and $D_{\mathcal{B}}$ and ρ are $\mathfrak{A}$-module maps. Therefore the assertions (i)-(v) hold. For (vi), suppose that $D_{\mathcal{A}}$, $D_{\mathcal{B}}$ and $\rho_{\mathcal{M}}$ are $\mathfrak{A}$-module maps. We show that D is a $\mathfrak{T}$-module map. Given $T_1 = \begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, T_2 = \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix} \in \mathcal{T}$ and $\Upsilon = \begin{bmatrix} \alpha & 0 \\ & \alpha \end{bmatrix} \in \mathfrak{T}$, we have

$$\begin{aligned}
\langle T_2, \Upsilon \cdot D(T_1)\rangle &= \langle T_2 \cdot \Upsilon, D(T_1)\rangle \\
&= \langle \begin{bmatrix} a_2 \cdot \alpha & m_2 \cdot \alpha \\ & b_2 \cdot \alpha \end{bmatrix}, \begin{bmatrix} D_{\mathcal{A}}(a_1) & \rho_{\mathcal{M}}(m_1) \\ & D_{\mathcal{B}}(b_1) \end{bmatrix}\rangle \\
&= \langle a_2 \cdot \alpha, D_{\mathcal{A}}(a_1)\rangle + \langle m_2 \cdot \alpha, \rho_{\mathcal{M}}(m_1)\rangle + \langle b_2 \cdot \alpha, D_{\mathcal{B}}(b_1)\rangle \\
&= \langle a_2, D_{\mathcal{A}}(\alpha \cdot a_1)\rangle + \langle m_2, \rho_{\mathcal{M}}(\alpha \cdot m_1)\rangle + \langle b_2, D_{\mathcal{B}}(\alpha \cdot b_1)\rangle \\
&= \langle \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}, \begin{bmatrix} D_{\mathcal{A}}(\alpha \cdot a_1) & \rho_{\mathcal{M}}(\alpha \cdot m_1) \\ & D_{\mathcal{B}}(\alpha \cdot b_1) \end{bmatrix}\rangle \\
&= \langle \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}, D\Big(\begin{bmatrix} \alpha & 0 \\ & \alpha \end{bmatrix} \cdot \begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}\Big)\rangle \\
&= \langle T_2, D(\Upsilon \cdot T_1)\rangle.
\end{aligned}$$

Therefore $D(\Upsilon \cdot T_1) = \Upsilon \cdot D(T_1)$. Similarly we obtain $D(T_1 \cdot \Upsilon) = D(T_1) \cdot \Upsilon$. Thus, D is a $\mathfrak{T}$-module map. □

The following sets which are used in this section are introduced in [13]. For each positive integer n, we define the centralizer of $\mathcal{A}$ in $\mathcal{A}^{(2n)}$ by $Z_{\mathcal{A}}(\mathcal{A}^{(2n)}) = \{x \in \mathcal{A}^{(2n)} : x \cdot a = a \cdot x \text{ for all } a \in \mathcal{A}\}$ and similarly, $Z_{\mathcal{B}}(\mathcal{B}^{(2n)}) = \{z \in \mathcal{B}^{(2n)} :$

$z \cdot b = b \cdot z$ for all $b \in \mathcal{B}\}$. The elements of

$$ZR_{\mathfrak{A},\mathcal{A},\mathcal{B}}(\mathcal{M},\mathcal{M}^{(2n)}) = \{\rho_{x,z} : \mathcal{M} \to \mathcal{M}^{(2n)} : x \in Z_{\mathcal{A}}(\mathcal{A}^{(2n)}), z \in Z_{\mathcal{B}}(\mathcal{B}^{(2n)})\}$$

are called *central Rosenblum operators* on $\mathcal{M}$ with coefficients in $\mathcal{M}^{(2n)}$ in which $\rho_{x,z}(m) = x \cdot m - m \cdot z$ is an $\mathfrak{A}$-module map. We also define the following set

$$\mathrm{Hom}_{\mathfrak{A},\mathcal{A},\mathcal{B}}(\mathcal{M},\mathcal{M}^{(2n)}) = \{\varphi : \mathcal{M} \to \mathcal{M}^{(2n)} : \varphi(a{\cdot}m) = a{\cdot}\varphi(m),\ \varphi(m{\cdot}b) = \varphi(m){\cdot}b,$$
$$\varphi(\alpha \cdot m) = \alpha \cdot \varphi(m),\ \varphi(m \cdot \alpha) = \varphi(m) \cdot \alpha,\ \text{for all } \alpha \in \mathfrak{A}, a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B}\}.$$

The following theorem is analogous to Lemma 3.11 and Theorem 3.12 from [13] in a more general setting (part (iii) is a module version of [16, Theorem 3.2]).

Theorem 5.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a unital Banach $\mathcal{A},\mathcal{B}$-module. Then we have*

(i) $ZR_{\mathfrak{A},\mathcal{A},\mathcal{B}}(\mathcal{M},\mathcal{M}^{(2n)}) \subseteq \mathrm{Hom}_{\mathfrak{A},\mathcal{A},\mathcal{B}}(\mathcal{M},\mathcal{M}^{(2n)})$,

(ii) *If* $\Omega \in \mathrm{Hom}_{\mathfrak{T},\mathcal{A},\mathcal{B}}\Big(\begin{bmatrix} 0 & \mathcal{M} \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{M}^{(2n)} \\ & 0 \end{bmatrix}\Big)$, *then the map*

$$\Delta_\Omega \begin{bmatrix} a & m \\ & b \end{bmatrix} = \Omega\Big(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} 0 & \Omega_{\mathcal{M}}(m) \\ & 0 \end{bmatrix} \in Z^1_{\mathfrak{T}}(\mathcal{T},\mathcal{T}^{(2n)}),$$

where $\Omega_{\mathcal{M}} \in \mathrm{Hom}_{\mathfrak{A},\mathcal{A},\mathcal{B}}(\mathcal{M},\mathcal{M}^{(2n)})$. Furthermore Δ_Ω is inner if and only if Ω is a central Rosenblum operator on $\mathcal{M}$ with coefficients in $\mathcal{M}^{(2n)}$.

(iii) *If $\mathcal{A}$ and $\mathcal{B}$ are $(2n)$-weakly module amenable as $\mathfrak{T}$-modules, then*

$$H^1_{\mathfrak{T}}(\mathcal{T},\mathcal{T}^{(2n)}) \simeq \mathrm{Hom}_{\mathfrak{T},\mathcal{A},\mathcal{B}}\Big(\begin{bmatrix} 0 & \mathcal{M} \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{M}^{(2n)} \\ & 0 \end{bmatrix}\Big)\Big/$$
$$ZR_{\mathfrak{T},\mathcal{A},\mathcal{B}}\Big(\begin{bmatrix} 0 & \mathcal{M} \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{M}^{(2n)} \\ & 0 \end{bmatrix}\Big).$$

In the other words,

$$H^1_{\mathfrak{T}}(\mathcal{T},\mathcal{T}^{(2n)}) \simeq \mathrm{Hom}_{\mathfrak{A},\mathcal{A},\mathcal{B}}(\mathcal{M},\mathcal{M}^{(2n)}) \big/ ZR_{\mathfrak{A},\mathcal{A},\mathcal{B}}(\mathcal{M},\mathcal{M}^{(2n)}).$$

Proof. For statements (i) and (ii), we only need to show that Δ_φ is an $\mathfrak{T}$-module map. The rest of the proof is the same as [13, Lemma 3.11]. For given $T = \begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$ and $\Upsilon = \begin{bmatrix} \alpha & 0 \\ & \alpha \end{bmatrix} \in \mathfrak{T}$, we have

$$\begin{aligned} \Delta_\Omega(\Upsilon \cdot T) &= \Delta_\Omega\Big(\begin{bmatrix} \alpha & 0 \\ & \alpha \end{bmatrix} \cdot \begin{bmatrix} a & m \\ & b \end{bmatrix}\Big) = \Delta_\Omega\Big(\begin{bmatrix} \alpha \cdot a & \alpha \cdot m \\ & \alpha \cdot b \end{bmatrix}\Big) \\ &= \begin{bmatrix} 0 & \alpha \cdot \Omega_{\mathcal{M}}(m) \\ & 0 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ & \alpha \end{bmatrix} \cdot \begin{bmatrix} 0 & \Omega_{\mathcal{M}}(m) \\ & 0 \end{bmatrix} = \Upsilon \cdot \Delta_\Omega(T), \end{aligned}$$

and similarly $\Delta_\Omega(T \cdot \Upsilon) = \Delta_\Omega(T) \cdot \Upsilon$.

For (iii), the proof is similar to [13, Theorem 3.12] but we give the proof for the sake of completeness. Consider

$$\mathfrak{F} : \mathrm{Hom}_{\mathfrak{T},\mathcal{A},\mathcal{B}}\Big(\begin{bmatrix} 0 & \mathcal{M} \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{M}^{(2n)} \\ & 0 \end{bmatrix} \Big) \to H^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n)})$$

defined by $\Omega \mapsto \overline{\Delta}_\Omega$, where $\overline{\Delta}_\Omega$ denotes the equivalence class of Δ_Ω in $H^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n)})$. It follows from (ii) that $\mathfrak{F}$ is an $\mathfrak{T}$-module map. Let $D : \mathcal{T} \longrightarrow \mathcal{T}^{(2n)}$ be a module derivation. By Lemma 5.1 there are module derivations $D_\mathcal{A} : \mathcal{A} \longrightarrow \mathcal{A}^{(2n)}$, $D_\mathcal{B} : \mathcal{B} \longrightarrow \mathcal{B}^{(2n)}$, and a $\mathfrak{A}$-module map $\rho : \mathcal{M} \longrightarrow \mathcal{M}^{(2n)}$ and an element $\gamma \in \mathcal{M}^{(2n)}$ such that

$$D\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix} \Big) = \begin{bmatrix} D_\mathcal{A}(a) & a \cdot \gamma - \gamma \cdot b + \rho(m) \\ & D_\mathcal{B}(b) \end{bmatrix}. \tag{5.1}$$

Since $\mathcal{A}$ and $\mathcal{B}$ both are $(2n)$-weakly module amenable, there exist $x \in \mathcal{A}$ and $y \in \mathcal{B}$ such that $D_\mathcal{A}(a) = a \cdot x - x \cdot a = D_x(a)$ and $D_\mathcal{B}(b) = b \cdot y - y \cdot b = D_y(b)$. Define $D_0 : \mathcal{T} \longrightarrow \mathcal{T}^{(2n)}$ as follows:

$$D_0\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix} \Big) = \begin{bmatrix} D_x(a) & a \cdot \gamma - \gamma \cdot b - \rho_{x,y}(m) \\ & D_y(b) \end{bmatrix}. \tag{5.2}$$

It is easy to see that D_0 is an inner $\mathfrak{T}$-module derivation induced by $\begin{bmatrix} x & \gamma \\ & y \end{bmatrix}$ (note that in the proof of [13, Theorem 3.12], there is a misprint). Set $D_1 = D - D_0$. Thus $\overline{D}_1 = \overline{D}$. Then

$$D_1\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix} \Big) = \begin{bmatrix} 0 & \rho(m) + \rho_{x,y}(m) \\ & 0 \end{bmatrix}, \tag{5.3}$$

for all $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$. Define $\Omega = \begin{bmatrix} 0 & \rho(m) + \rho_{x,y}(m) \\ & 0 \end{bmatrix}$. Clearly Ω belongs to

$$\mathrm{Hom}_{\mathfrak{T},\mathcal{A},\mathcal{B}}(\begin{bmatrix} 0 & \mathcal{M} \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{M}^{(2n)} \\ & 0 \end{bmatrix}).$$

By (ii), we have $\mathfrak{F}(\Omega) = \overline{\Delta}_\Omega = \overline{D}_1 = \overline{D}$. This means that $\mathfrak{F}$ is surjective. Finally, we must show that $\ker \mathfrak{F} = ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})$. Let $\Omega \in \ker \mathfrak{F}$. Then $\mathfrak{F}(\Omega)\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix} \Big) = \overline{D}\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix} \Big) = 0 = \overline{\Delta}_\Omega\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix} \Big)$, hence $\overline{\Delta}_\Omega$ is inner. Again by (ii), we have $\Omega \in ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})$, and (i) implies that

$\ker \mathfrak{F} = ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})$. Therefore

$$\begin{aligned} H^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n)}) &\simeq \mathrm{Hom}_{\mathfrak{T},\mathcal{A},\mathcal{B}}\Big(\begin{bmatrix} 0 & \mathcal{M} \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{M}^{(2n)} \\ & 0 \end{bmatrix}\Big)\Big/ \ker \mathfrak{F} \\ &= \frac{\mathrm{Hom}_{\mathfrak{T},\mathcal{A},\mathcal{B}}\Big(\begin{bmatrix} 0 & \mathcal{M} \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{M}^{(2n)} \\ & 0 \end{bmatrix}\Big)}{ZR_{\mathfrak{T},\mathcal{A},\mathcal{B}}\Big(\begin{bmatrix} 0 & \mathcal{M} \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{M}^{(2n)} \\ & 0 \end{bmatrix}\Big)}. \end{aligned}$$

□

The following result may be proved like Proposition 4.2 using a modification of Lemma 5.1.

Proposition 5.2. *Let $\mathcal{A}$ or $\mathcal{B}$ has a bounded approximate identity, and let $\mathcal{A}^{(2n)}$, $\mathcal{B}^{(2n)}$ and $\mathcal{M}^{(2n)}$ be non-degenerate. Then the $\mathfrak{T}$-module map $D : \mathcal{T} \longrightarrow \mathcal{T}^{(2n)}$ is a module derivation if and only if there exist module derivations $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^{(2n)}$ and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}^{(2n)}$, and $\mathfrak{A}$-module map $\rho : \mathcal{M} \longrightarrow \mathcal{M}^{(2n)}$ which satisfies conditions* (iv) *and* (v) *of Lemma 5.1 such that*

$$D\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\Big) = \begin{bmatrix} D_{\mathcal{A}}(a) & a \cdot \gamma - \gamma \cdot b + \rho(m) \\ & D_{\mathcal{B}}(b) \end{bmatrix}, \tag{5.4}$$

in which $\gamma \in \mathcal{M}^{(2n)}$ and $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$.

Proof. We argue similar to the proof of Proposition 4.2. Let $D : \mathcal{T} \longrightarrow \mathcal{T}^{(2n)}$ be a continuous module derivation. Define $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^{(2n)}$ by $D_{\mathcal{A}}(a) = \pi_{\mathcal{A}}(D\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big))$, and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}^{(2n)}$ by $D_{\mathcal{B}}(b) = \pi_{\mathcal{B}}(D\Big(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\Big))$. Obviously these maps are $\mathfrak{A}$-module maps, and by [16, Lemma 2.3], they are module derivations. Let $(e_\alpha)_{\alpha \in \Lambda}$ be a bounded approximate identity of $\mathcal{A}$, and let $D\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} D_{\mathcal{A}}(a) & \eta \\ & \mu \end{bmatrix}$ for all $a \in \mathcal{A}$. Then

$$\begin{aligned} D\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big) &= D\Big(\begin{bmatrix} \lim_\alpha a e_\alpha & 0 \\ & 0 \end{bmatrix}\Big) = D\Big(\lim_\alpha \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} e_\alpha & 0 \\ & 0 \end{bmatrix}\Big) \\ &= \lim_\alpha \Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix} \cdot \begin{bmatrix} D_{\mathcal{A}}(e_\alpha) & \gamma \\ & \theta \end{bmatrix} \\ &+ \begin{bmatrix} D_{\mathcal{A}}(a) & \eta \\ & \mu \end{bmatrix} \cdot \begin{bmatrix} e_\alpha & 0 \\ & 0 \end{bmatrix}\Big) \\ &= \begin{bmatrix} \lim_\alpha D_{\mathcal{A}}(a e_\alpha) & a \cdot \gamma \\ & 0 \end{bmatrix} = \begin{bmatrix} D_{\mathcal{A}}(a) & a \cdot \gamma \\ & 0 \end{bmatrix}. \end{aligned} \tag{5.5}$$

Let $b \in \mathcal{B}$ and $D\Big(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\Big) = \begin{bmatrix} \theta & \eta \\ & D_{\mathcal{B}}(b) \end{bmatrix}$. Since $\mathcal{M}^{(2n)}$ and $\mathcal{A}^{(2n)}$ are non-degenerate (see [16, Theorem 3.2]), we have

$$D\Big(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\Big) = \begin{bmatrix} 0 & -\gamma \cdot b \\ & D_{\mathcal{B}}(b) \end{bmatrix}. \tag{5.6}$$

For $m \in \mathcal{M}$, suppose that $D\Big(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} \theta & \eta \\ & \mu \end{bmatrix}$, then by [16, Theorem 3.2], $\theta = \mu = 0$. Now, we define $\rho : \mathcal{M} \longrightarrow \mathcal{M}^{(2n)}$ by $\rho(m) = \pi_{\mathcal{M}}(D\Big(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\Big))$. It is clear that ρ is an $\mathfrak{A}$-module map. Therefore it suffices to show that ρ satisfies in conditions (iv) and (v) of Lemma 5.1. We check these conditions as follows:

$$\begin{aligned}
\rho(a \cdot m) &= \pi_{\mathcal{M}}(D\Big(\begin{bmatrix} 0 & a \cdot m \\ & 0 \end{bmatrix}\Big)) = \pi_{\mathcal{M}}(D\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\Big)) \\
&= \pi_{\mathcal{M}}\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \eta \\ & 0 \end{bmatrix} + \begin{bmatrix} D_{\mathcal{A}}(a) & a \cdot \gamma \\ & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\Big) \\
&= a \cdot \eta + D_{\mathcal{A}}(a) \cdot m = a \cdot \rho(m) + D_{\mathcal{A}}(a) \cdot m,
\end{aligned}$$

and similarly we have $\rho(m \cdot b) = \rho(m) \cdot b + m \cdot D_{\mathcal{B}}(b)$. □

We can now rephrase part (iii) of Theorem 5.1 as follows.

Theorem 5.3. *Let $\mathcal{A}$ or $\mathcal{B}$ has a bounded approximate identity, and let $\mathcal{A}^{(2n)}$, $\mathcal{B}^{(2n)}$ and $\mathcal{M}^{(2n)}$ be non-degenerate. If $\mathcal{A}$ and $\mathcal{B}$ are $(2n)$-weakly module amenable as $\mathfrak{A}$-modules, then*

$$H^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n)}) \simeq \mathrm{Hom}_{\mathfrak{A},\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}) / ZR_{\mathfrak{A},\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}). \tag{5.7}$$

Proof. Appling Proposition 5.2, the argument of Theorem 5.1 can be repeated to obtain the result. □

Corollary 5.3.1. *Let $\mathcal{A}$ has a bounded approximate identity, and $\mathcal{A}^{(2n)}$ be non-degenerate. If $\mathcal{A}$ is $(2n)$-weakly module amenable (as an $\mathfrak{A}$-module), then $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ & \mathcal{A} \end{bmatrix}$ is $(2n)$-weakly module amenable (as an $\mathfrak{T}$-module).*

Proof. Let (e_α) be a bounded approximate identity of $\mathcal{A}$, and let φ in $\mathrm{Hom}_{\mathfrak{A},\mathcal{A},\mathcal{A}}$ $(\mathcal{A}, \mathcal{A}^{(2n)})$. Then there exists $E \in \mathcal{A}^{(2n)}$ and a subnet $\{\varphi(e_\beta)\}$ of $\{\varphi(e_\beta)\}$ such that $\varphi(e_\beta) \xrightarrow{w^*} E$. We have

$$\varphi(a) = w^* - \lim_{\beta} \varphi(e_\beta a) = w^* - \lim_{\beta} e_\beta \varphi(a) = Ea.$$

Similarly, $\varphi(a) = aE$. This shows that $\varphi \in ZR_{\mathfrak{A},\mathcal{A},\mathcal{A}}(\mathcal{A}, \mathcal{A}^{(2n)})$ and $H^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n)})$ $= \{0\}$ by Theorem 5.3. □

6. Examples

In this section, by using the results of the previous sections we show that under which conditions the Banach algebra $\mathcal{T}_0 = \begin{bmatrix} \ell^1(S) & \ell^1(S) \\ & \ell^1(S) \end{bmatrix}$ is permanently weakly module amenable and module Arens regular where S is an inverse semigroup.

Definition 6.1. A discrete semigroup S is called an inverse semigroup if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotents of S is denoted by E.

Let S be an inverse semigroup with the set of idempotents E. By [14, Theorem V.1.2] E is a commutative subsemigroup of S and a semilattice, $\ell^1(E)$ could be regarded as a commutative subalgebra of $\ell^1(S)$, and therefore $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible actions [1].

Let $k \in \mathbb{N}$. Recall that E satisfies condition D_k [11] if given $f_1, f_2, ..., f_{k+1} \in E$ there exist $e \in E$ and i, j such that

$$1 \leq i < j \leq k+1, f_i e = f_i, f_j e = f_j.$$

Duncan and Namioka in [11, Theorem 16] proved that for any inverse semigroup S, $\ell^1(S)$ has a bounded approximate identity if and only if E satisfies condition D_k for some k.

Let S be a commutative inverse semigroup with the set of idempotents E. Consider $\ell^1(S)$ as an $\ell^1(E)$-module with the following action:

$$\delta_e \cdot \delta_s = \delta_s \cdot \delta_e = \delta_s * \delta_e = \delta_{se}, \quad (s \in S, e \in E). \tag{6.1}$$

Theorem 6.1. *Let $n \in \mathbb{N}$ and let S be a commutative inverse semigroup with the set of idempotents E. Then $\ell^1(S)$ is n-weakly module amenable as an $\ell^1(E)$-module with the actions* (6.1).

Proof. For any semigroup S, group algebra $\ell^1(S)$ is commutative if and only if S is commutative. Since $\ell^1(S)$ is a bi-commutative Banach $\ell^1(S)$-$\ell^1(E)$-module, so is $\ell^1(S)^{(n)}$. By [4, Theorem 3.1], $\ell^1(S)$ is weakly module amenable as an $\ell^1(E)$-module. The semigroup algebra $\ell^1(S)$ is essential, in fact $\ell^1(S) = \ell^1(S) \star \ell^1(E) \subseteq \ell^1(S) \star \ell^1(S) \subseteq \ell^1(S)$ (see the proof of [6, Theorem 3.15]). Now, it follows from Lemma 4.1 that every module derivation from $\ell^1(S)$ into $\ell^1(S)^{(n)}$ is zero. This shows that $\ell^1(S)$ is n-weakly module amenable. $\square$

In [18], the authors proved that the Banach algebra $\mathcal{T}_0 = \begin{bmatrix} \ell^1(S) & \ell^1(S) \\ & \ell^1(S) \end{bmatrix}$ is weak $\mathfrak{T}_0$-module amenable in which $\mathfrak{T}_0 := \left\{ \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \mid \alpha \in \ell^1(E) \right\}$, where S

is a unital commutative inverse semigroup. The condition of being unital for S is rather strong and it can be replaced by the weaker condition that E satisfies condition D_k for some k.

Example 1. Let S be a commutative inverse semigroup such that E satisfies condition D_k for some k. By Theorem 4.3 we have

$$\mathcal{H}^1_{\mathfrak{T}_0}(\mathcal{T}_0, \mathcal{T}_0^{(2n-1)}) \simeq \mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(2n-1)}) \oplus \mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(2n-1)}).$$

where $\mathcal{T}_0$ is a Banach $\mathfrak{T}_0$-module. Since $\ell^1(S)$ is n-weakly module amenable (Theorem 6.1), $\mathcal{T}_0$ is $(2n+1)$-weakly module amenable (as $\mathfrak{T}_0$-module) again by Theorem 4.3. On the other hand, $\ell^1(S)$ possess a bounded approximate identity, and thus $\mathcal{T}_0$ is $(2n)$-weakly module amenable by Corollary 5.3.1. Therefore $\mathcal{T}_0$ is permanently weakly module amenable.

Here, for technical reasons, we let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e \cdot \delta_s = \delta_s, \ \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \quad (s \in S, e \in E).$$

In this case, the ideal J (see section 2) is the closed linear span of $\{\delta_{set} - \delta_{st} \quad s, t \in S, e \in E\}$. We consider an equivalence relation on S as follows:

$$s \approx t \Longleftrightarrow \delta_s - \delta_t \in J \ \ (s, t \in S).$$

For an inverse semigroup S, the quotient $S/\approx$ is a discrete group (see [2] and [19]). Indeed, $S/\approx$ is homomorphic to the maximal group homomorphic image G_S [17] of S [20]. In particular, S is amenable if and only if G_S is amenable [11, 17].

Example 2. The Banach algebra $\mathcal{T}_0$ is module Arens regular (as an Banach $\mathfrak{T}_0$-module) if and only if $\ell^1(S)$ is module Arens regular as an $\ell^1(E)$-module with trivial left action and canonical right action by Theorem 3.1. Now it follows from [22, Theorem 3.3] that $\mathcal{T}_0$ is module Arens regular if and only if is G_S is finite.

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* Department of Mathematics
Garmsar Branch
Islamic Azad University
Garmsar, IRAN
E-mail address: abasalt.bodaghi@gmail.com

** Young Researchers Club
Ardabil Branch
Islamic Azad University
Ardabil, IRAN
E-mail address: jabbari_al@yahoo.com